

# On Powers of Some Power Series

Milan Janjić

Department of Mathematics and Informatics  
 University of Banja Luka  
 78000 Banja Luka, Republic of Srpska, BA  
 email: agnus@blic.net

## Abstract

We investigate some relationships between powers of powers series and compositions of positive integers. We show that the compositions may be interpreted in terms of powers of some power series, over arbitrary commutative ring. As consequences, several closed formulas for the compositions as well as for the generalized compositions with a fixed number of parts are derived. Some results on compositions obtained in some recent papers are consequences of these formulas.

Keywords: Compositions, Power Series.

## 1 Introduction

Let  $R$  be a commutative ring with 1, and let  $\mathbf{r} = (r_0, r_1, \dots)$  be any sequence of elements of  $R$ . Let  $k, n$  be nonnegative integers such that  $k \leq n$ . We define the finite sequence  $C^{(\mathbf{r})}(n, k)$ ,  $(k = 0, 1, \dots, n)$  of elements of  $R$  in the following way:

$$C^{(\mathbf{r})}(n, 1) = r_{n-1}, C^{(\mathbf{r})}(0, 0) = 1, C^{(\mathbf{r})}(n, 0) = C^{(\mathbf{r})}(n, k) = 0, (0 < n < k),$$

and,

$$C^{(\mathbf{r})}(n, k) = \sum_{i=0}^{n-k} r_i C^{(\mathbf{r})}(n-i-1, k-1), (1 < k \leq n). \quad (1)$$

We also define the sequence  $C^{(\mathbf{r})}(n)$ ,  $(n = 0, 1, 2, \dots)$  to be

$$C^{(\mathbf{r})}(0) = 1, C^{(\mathbf{r})}(n) = \sum_{k=1}^n C^{(\mathbf{r})}(n, k), (n \geq 1). \quad (2)$$

For a fixed positive integer  $k$  we let  $\binom{\mathbf{r}, k}{n}$ ,  $(n = 0, 1, \dots)$  denote the sequence which a generating function is  $(\sum_{i=0}^{\infty} r_i x^i)^k$ . Thus,

$$\left( \sum_{i=0}^{\infty} r_i x^i \right)^k = \sum_{n=0}^{\infty} \binom{\mathbf{r}, k}{n} x^n. \quad (3)$$

When the  $b$ 's are nonnegative integers, then  $C^{(\mathbf{r})}(n, k)$  is the number of generalized compositions of  $n$  into  $k$  parts [M2], Theorem 1). Also, It is proved in [M1] that, in this case,  $C^{(\mathbf{r})}(n)$  is the number of all generalized compositions of  $n$ .

In Section 2 we derive a recurrence equation for  $C^{(\mathbf{r})}(n)$ . Further we find a generating function for  $C^{(\mathbf{r})}(n, k)$ . In Section 3 we apply the results obtained in Section 2, on usual compositions. Finally, in Section 4 we derive some results for generalized compositions.

## 2 A generating function

**Proposition 1.** *Let  $n$  be a positive integer, and let  $\mathbf{r} = (r_0, r_1, \dots)$  be a sequence in  $R$ . Then*

$$C^{(\mathbf{r})}(n) = \sum_{i=0}^{n-1} r_i C^{(\mathbf{r})}(n-i-1). \quad (4)$$

*Proof.* We use induction on  $n$ . For  $n = 1$  the assertion is clearly true. Suppose that it is true for  $k < n$ . We have

$$C^{(\mathbf{r})}(n) = \sum_{k=1}^n C^{(\mathbf{r})}(n, k) = \sum_{k=1}^n \sum_{i=0}^{n-k} r_i C^{(\mathbf{r})}(n-i-1, k-1).$$

Changing the order of summation we obtain

$$C^{(\mathbf{r})}(n) = r_{n-1} + \sum_{i=0}^{n-2} \sum_{k=1}^{n-i} r_i C^{(\mathbf{r})}(n-i-1, k-1).$$

Taking into account that  $C^{(\mathbf{r})}(m, 0) = 0$  if  $m > 0$ , and then applying the induction hypothesis we get

$$C^{(\mathbf{r})}(n) = r_{n-1} + \sum_{i=0}^{n-2} r_i C^{(\mathbf{r})}(n-i-1),$$

that is

$$C^{(\mathbf{r})}(n) = \sum_{i=0}^{n-1} r_i C^{(\mathbf{r})}(n-i-1),$$

and the proposition is true.  $\square$

Next, for a positive integer  $k$ , by  $\binom{\mathbf{r}, k}{n}$ , ( $n = 0, 1, \dots$ ) we denote the sequence which a generating function is

$$g(x, k) = \left( \sum_{i=0}^{\infty} r_i x^i \right)^k. \quad (5)$$

Hence,

$$\left( \sum_{i=0}^{\infty} r_i x^i \right)^k = \sum_{n=0}^{\infty} \binom{\mathbf{r}, k}{n} x^n. \quad (6)$$

Thus,

$$\binom{\mathbf{r}, k}{n} = \sum_{i_1+i_2+\dots+i_k=n} r_{i_1} r_{i_2} \cdots r_{i_k}, \quad (7)$$

where the sum is taken over  $i_t \geq 0$ , ( $i = 1, 2, \dots, k$ ).

**Proposition 2.** *Let  $n, k$  be nonnegative integers such that  $k \leq n$ . Then,*

$$C^{(\mathbf{r})}(n, k) = \binom{\mathbf{r}, k}{n - k}. \quad (8)$$

*Proof.* For  $n = k$  we have  $C^{(\mathbf{r})}(k, k) = \binom{\mathbf{r}, 0}{k} = r_0^k$ , and the assertion is true. Assume that  $k < n$ . We shall prove the proposition by induction on  $k$ .

$$C^{(\mathbf{r})}(n, k) = \binom{\mathbf{r}, k}{n - k}. \quad (9)$$

For  $k = 1$  equation (6) takes the form:

$$r_0 + r_1 x + \cdots = \binom{\mathbf{r}, 1}{0} + \binom{\mathbf{r}, 1}{1} x + \cdots.$$

Since  $C^{(\mathbf{r})}(n, 1) = r_{n-1}$ , we have

$$C^{(\mathbf{r})}(n, 1) = \binom{\mathbf{r}, 1}{n - 1}, \quad (1 \leq n),$$

and the assertion is true for  $k = 1$ .

Assume that  $k > 1$ , and that the assertion is true for  $k - 1$ . We obviously have

$$\left( \sum_{i=0}^{\infty} r_i x^i \right)^k = \left( \sum_{i=0}^{\infty} r_i x^i \right) \cdot \left( \sum_{i=0}^{\infty} r_i x^i \right)^{k-1}.$$

Using (6) we may write this equation in the form:

$$\sum_{i=0}^{\infty} \binom{k, \mathbf{r}}{i} x^i = \sum_{i=0}^{\infty} r_i x^i \cdot \sum_{i=0}^{\infty} \binom{\mathbf{r}, k-1}{i} x^i,$$

so that,

$$\sum_{i=0}^{\infty} \binom{\mathbf{r}, k}{i} x^i = \sum_{i=0}^{\infty} \sum_{j=0}^i r_j \binom{\mathbf{r}, k-1}{i-j} x^i.$$

By comparing the terms of  $x^{n-k}$ , in the preceding equation, we obtain

$$\binom{k, \mathbf{r}}{n - k} = \sum_{j=0}^{n-k} r_j \binom{\mathbf{r}, k-1}{n - k - j}.$$

Using the induction hypothesis we obtain

$$\binom{k, \mathbf{r}}{n-k} = \sum_{j=0}^{n-k} r_j C^{(\mathbf{r})}(n-j-1, k-1),$$

and the assertion follows from (1).  $\square$

As an immediate consequence we have

$$C^{(\mathbf{r})}(n) = \sum_{k=1}^n \binom{\mathbf{r}, k}{n-k}. \quad (10)$$

### 3 Some results on compositions

In this section we assume that each  $b_i$  is either 1 or 0. It is proved in [M3] that then  $C^{(\mathbf{r})}(n, k)$  counts the number of compositions of  $n$  into  $k$  parts, all of which are in the set  $\{i : b_i = 1\}$ . Also, in this case the generating function  $g(x, k)$  is similar to the function defined in [HS]. The number  $C^{(\mathbf{r})}(n)$  counts the number of all such compositions.

**Proposition 3.** 1. If  $r_i = 1$ , ( $i = 0, 1, \dots$ ), then  $\binom{\mathbf{r}, k}{n}$  count the number of nonnegative solutions of the Diophant equation

$$i_1 + i_2 + \dots + i_k = n.$$

Hence,

$$\binom{\mathbf{r}, k}{n} = \binom{n+k-1}{n}.$$

2. Assume that there is a positive number  $m$  such that  $r_0 = r_1 = \dots = r_{r-1} = 1$ ,  $r_i = 0$ , ( $i > m$ ). Then  $\binom{\mathbf{r}, k}{n}$  counts the number of nonnegative solutions of the Diophant equation

$$i_1 + i_2 + \dots + i_k = n,$$

in which each term is  $\leq m$ .

*Proof.* 1. The first assertion follows from (7). The second assertion follows from the well-known fact that the Diophant equation in 1. has exactly  $\binom{n+k-1}{n}$  solutions.

2. The assertion follows from (7).  $\square$

In the next result we prove that, when each part of a composition belong to a fixed finite set, containing 1, then the number of compositions is a sum of multinomial coefficients. The case when 1 does not belong to the set is easily reduced to this case.

**Proposition 4.** Assume that  $r_i = 1$  if  $i \in \{1, m_1, \dots, m_s\}$ , and  $r_i = 0$  otherwise. Then

$$C^{(\mathbf{r})}(n, k) = \sum_{(i_0, i_1, \dots, i_s)} \binom{k}{i_0, i_1, \dots, i_s},$$

where the sum on the right-hand side is taken over all nonnegative solutions of the Diophant equations

$$i_0 + i_1 + \dots + i_s = k, \quad (m_1 - 1)i_1 + (m_2 - 1)i_2 + \dots + (m_s - 1)i_s = n - k.$$

*Proof.* In this case, the formula (10) becomes

$$g(x, k) = (1 + x^{m_1-1} + x^{m_2-1} + \dots + x^{m_s-1})^k.$$

Using the multinomial formula yields

$$g(x, k) = \sum_{(i_0, i_1, \dots, i_s)} \binom{k}{i_0, i_1, \dots, i_s} x^{(m_1-1)i_1 + \dots + (m_s-1)i_s},$$

where the sum on the right-hand side is taken over all nonnegative solutions of the Diophant equation

$$i_0 + i_1 + \dots + i_s = k.$$

and the proposition follows from Proposition 2.  $\square$

The simplest case is  $s = 1$ ,  $r_1 = 2$ .

**Corollary 1.** The number of the compositions of  $n$  with  $k$  parts, each equals either 1 or 2 is  $\binom{k}{n-k}$ .

Next, suppose that  $s = 1$ ,  $r_1 = m > 2$ . Then,

$$C^{(\mathbf{r})}(n, k) = \sum_{(i_0, i_1)} \binom{k}{i_0, i_1},$$

where  $i_0 + i_1 = k$ ,  $(m - 1)i_1 = n - k$ . The last equation has a solution if and only if  $m - 1$  divides  $n - k$ .

We thus obtain the closed formulas for the number of compositions into  $k$  parts and for the number of all compositions.

**Corollary 2.** Let  $n, k, m$  be positive integers with  $m > 1$ . For the number  $C^{(\mathbf{r})}(n, k)$  of compositions of  $n$  with  $k$  parts, all of which are either 1 or  $m$  we have is

$$C^{(\mathbf{r})}(n, k) = \binom{k}{\frac{n-k}{m-1}},$$

if  $m - 1$  divides  $n - k$ , and 0 otherwise. For the number of all such compositions we have

$$C^{(\mathbf{r})}(n) = \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \binom{n - (m-1)j}{j}.$$

**Remark 1.** This kind of compositions is investigated in [C1].

**Proposition 5.** Let  $n, m$  be positive integers, and let  $r_i = 1$  if  $i = 1 \pmod{m}$ , and  $r_i = 0$  otherwise. Then

$$C^{(\mathbf{r})}(n, k) = \binom{\frac{n-k}{m} + k - 1}{\frac{n-k}{m}}, \text{ if } n = k \pmod{m},$$

and  $C^{(\mathbf{r})}(n, k) = 0$ , otherwise.

*Proof.* According to (3), for the usual composition the formula (6) has the form

$$\left( \sum_{i=0}^{\infty} x^i \right)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n. \quad (11)$$

The proposition follows by replacing  $x$  by  $x^m$ .  $\square$

**Corollary 3.** Let  $n$  be a positive integer. Then,

$$F_n = \sum_k \binom{\frac{n-k}{2} + k - 1}{\frac{n-k}{2}},$$

where the sum is taken over  $k$ , such that  $k$  and  $n$  are of the same parity.

*Proof.* For  $m = 2$ , the numbers  $C^{(\mathbf{r})}(n, k)$  from the preceding proposition count the numbers of compositions with odd parts. Hence,  $C^{(\mathbf{r})}(n, k) = \binom{\frac{n-k}{2} + k - 1}{k-1}$ , if  $n$  and  $k$  are of the same parity, and  $C^{(\mathbf{r})}(n, k) = 0$ , otherwise. On the other hand, it is proved in Corollary 26, in [M1], that the number of all compositions of  $n$  is  $F_n$ .  $\square$

In the paper [C3] the compositions without part equals 2 are considered. We have derived a closed formula for the number of such compositions with  $k$  parts. From the formula all results from Theorem 4 to Theorem 9, in [C3], may be derived.

**Proposition 6.** Let  $n, k$  be positive integers such that  $k \leq n$ . If  $c_k(n)$  is the number of compositions of  $n$  with  $k$  parts, all different of 2, then

$$c_n(n) = 1, \quad c_{n-1}(n) = 0, \quad c_k(n) = \sum_{i=1}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k}{i} \binom{n-k-i-1}{i-1}, \quad (k < n-1).$$

*Proof.* Equation (6), in this case, take the form:

$$(1 + x^2 + x^3 + \dots)^k = \sum_{n=0}^{\infty} \binom{\mathbf{r}, k}{n} x^n.$$

Using the binomial theorem and (11) we obtain

$$1 + \sum_{i=1}^k \sum_{j=0}^{\infty} \binom{k}{i} \binom{j+i-1}{j} x^{2i+j} = \sum_{n=0}^{\infty} \binom{\mathbf{r}, k}{n} x^n.$$

It follows that

$$\binom{\mathbf{r}, k}{0} = 1, \quad \binom{\mathbf{r}, k}{1} = 0, \quad \binom{\mathbf{r}, k}{n} = \sum_{2i+j=n} \binom{k}{i} \binom{j+i-1}{j}, \quad (k > 1).$$

The propositions now follows from Proposition 2.  $\square$

**Corollary 4** (Theorem 9, [C3]). *The number  $C_k(k+8)$  of  $k+8$  into  $k$  parts, all of which are different from 2, is*

$$c_k(k+8) = k + 5 \binom{k}{2} + 6 \binom{k}{3} + \binom{k}{4}.$$

*Proof.* In this case we have

$$c_k(k+8) = \sum_{i=1}^4 \binom{k}{i} \binom{8-i-1}{i-1},$$

and the corollary is true.  $\square$

Similarly, we obtain the formula for the compositions without parts equal  $m$ , for any  $m$ .

**Proposition 7.** *Let  $n, k$  be positive integer such that  $k \leq n$ , let  $m$  be a non-negative integer, and  $b_m = 0$ ,  $b_i = 1$ ,  $(i \neq m)$ .*

$$C^{(\mathbf{r})}(n, k) = \sum_{i=0}^k \sum_{i_0+i_1+\dots+i_{m-1}=k-i} \binom{k}{i} \binom{k-i}{i_0, i_1, \dots, i_{m-1}} \binom{N}{i-1}, \quad (12)$$

where  $N = n - k - ri - 1 - i_1 - \dots - (m-1)i_{m-1}$ .

*Proof.* We consider here the compositions without parts equal  $m+1$ . The generating function for the number of such compositions with  $k$  parts is

$$g(x, k) = [1 + x + \dots + x^{m-1} + x^{m+1}(1 + x + x^2 + \dots)]^k,$$

that is,

$$g(x, k) = \sum_{i=0}^k \binom{k}{i} (1 + x + \dots + x^{m-1})^{k-i} (1 + x + \dots)^i x^{(m+1)i}.$$

Using (11) and the multinomial formula yields

$$g(x, k) = \sum_{i=0}^k \sum_{j=0}^{\infty} \sum_{i_0+i_1+\dots+i_{m-1}=k-i}^{\infty} X,$$

where

$$X = \binom{k}{i} \binom{k-i}{i_0, i_1, \dots, i_{m-1}} \binom{j+i-1}{j} x^{(m+1)i+j+i_1+2i_2+\dots+(m-1)i_{m-1}}.$$

It follows that

$$\binom{\mathbf{r}, k}{n} = \sum_{i=0}^k \sum_{i_0+i_1+\dots+i_{m-1}=k-i} \binom{k}{i} \binom{k-i}{i_0, i_1, \dots, i_{m-1}} \binom{N}{i-1},$$

where  $N = n - mi - 1 - i_1 - \dots - (m-1)i_{m-1}$ , and the proposition follows.  $\square$

From the preceding proposition we may obtain several closed formula for the numbers of compositions considered in Section 5, [C2]. One of them is the following:

**Corollary 5.** *Let  $n, k$  be positive integers such that  $k \leq n$ . Then the number  $c_k^{(3)}(n)$  of the compositions of  $n$  into  $k$  parts with no occurrence of 3 is*

$$c_k^{(3)}(n) = \sum_{j=0}^k \sum_{i=0}^{\min\{k-j, \lfloor \frac{n-2k+j}{2} \rfloor\}} \binom{k}{i} \binom{k-i}{j} \binom{n-2k+j-i-1}{i-1}.$$

*Proof.* This is the case when  $m = 2$ . Equation (12) becomes

$$c_k^{(3)}(n) = \sum_{i=0}^k \sum_{i_0+i_1=k-i} \binom{k}{i} \binom{k-i}{i_0, i_1} \binom{n-k-2i-1-i_1}{i-1},$$

that is,

$$c_k^{(3)}(n) = \sum_{i=0}^k \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \binom{n-2k+j-i-1}{i-1},$$

if the conditions  $n - 2k + j - i - 1 \geq i - 1$  is fulfilled. Changing the order of the summation and using the preceding condition we conclude that the corollary is true.  $\square$

## 4 Some results for generalized compositions

It is proved in [M3] that in the case when each  $r_i$  is a nonnegative integer, then  $C^{(\mathbf{r})}(n, k)$  is the number of the generalized compositions of  $n$  with  $k$  parts. Also,  $C^{(\mathbf{r})}(n)$  is the number of all generalized compositions of  $n$ . In the paper [GR],

the author considered the compositions when there are two types of one, and one type of all other natural numbers. But, no a formula for the number of such compositions with a fixed number of parts is derived. We start this section with such a formula, but in a more general case when there are any number of ones.

**Proposition 8.** *Let  $n, k$  be nonnegative integers such that  $k \leq n$ , let  $m$  be a nonnegative integer, and let  $r_0 = 1 + m$ ,  $r_i = 1$ , ( $i > 0$ ). Then,*

$$C^{(\mathbf{r})}(n, n) = (1 + m)^k, \quad C^{(\mathbf{r})}(n, k) = \sum_{i=1}^k \binom{k}{i} \binom{n - k + i - 1}{i - 1} m^{k-i}, \quad (k < n).$$

*Proof.* In this case we have

$$g(k, x) = (m + 1 + x + x^2 + \cdots)^k.$$

Using binomial formula we obtain

$$g(k, x) = \sum_{i=0}^k \binom{k}{i} m^{k-i} (1 + x + x^2 + \cdots)^i.$$

Equation xx implies

$$g(k, x) = m^k + \sum_{i=1}^k \sum_{j=0}^{\infty} \binom{k}{i} \binom{j + i - 1}{j} m^{k-i} x^j.$$

It follows that

$$\binom{\mathbf{r}, k}{0} = (1 + m)^k, \quad \binom{\mathbf{r}, k}{n} = \sum_{i=1}^k \binom{k}{i} \binom{n + i - 1}{i - 1} m^{k-i}, \quad (n \geq 1),$$

and the proposition is true.

In particular, if  $m = 1$ , the in [GR] is proved that  $F_{2n+1}$  is the number of all compositions of this kind. Thus,

**Corollary 6.** *The following identity is true*

$$F_{2n+1} = 2^n + \sum_{i=1}^{n-1} \binom{k}{i} \binom{n - k - i - 1}{i - 1},$$

where  $F_{2n+1}$  is the Fibonacci number.

□

**Proposition 9.** *Let  $n, k, m$  be positive integer such that  $k \leq n$ , and let  $r_i = \binom{m}{i}$ , ( $i = 0, 1, \dots$ ). Then,*

$$C^{(\mathbf{r})}(n, k) = \binom{mk}{n - k}.$$

*Proof.* We have

$$g(k, x) = \left( \sum_{i=0}^m \binom{m}{i} x^i \right)^k.$$

Using the binomial formula implies

$$g(k, x) = (1 + x)^{mk},$$

and the proposition is true.  $\square$

In the next result we show that when the  $b$ 's are pyramidal numbers then the number of compositions is a pyramidal number also.

**Proposition 10.** *Let  $n, k, m$  be positive integer such that  $k \leq n$ , and let  $r_i = \binom{i+m-1}{i}$ , ( $i = 0, 1, \dots$ ). Then,*

$$C^{(\mathbf{r})}(n, k) = \binom{n - k + mk - 1}{n - k}.$$

*Proof.* From (11) follows

$$\left[ \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n \right]^m = \left( \sum_{i=0}^{\infty} x^i \right)^{km},$$

and the proposition is true according to (6).  $\square$

In the case  $m = 2$  we have  $r_i = i + 1$ , ( $i = 0, 1, \dots$ ), so that,

$$C^{(\mathbf{r})}(n, k) = \binom{n+k-1}{2k-1},$$

which is the result of Theorem 1, [AG].

**Proposition 11.** *Let  $n, k$  be positive integers such that  $k \leq n$ , and let  $r_i = (i + 1)^2$ , ( $i = 0, 1, \dots$ ). Then*

$$C^{(\mathbf{r})}(n, k) = \sum_{i=0}^{n-k} \binom{k}{i} \binom{n-k+2i-1}{3i-1}.$$

*Proof.* It is a well-known that the generating function  $f(x)$  of the sequence  $1, 2^2, 3^2, \dots$  is

$$f(x) = (1 + x)(1 + x + x^2 + \dots)^3.$$

It follows that

$$[f(x)]^k = \sum_{i=0}^{\infty} \sum_{j=0}^i \binom{k}{j} \binom{i-j+3k-1}{i-j} x^i.$$

According to (6) we conclude that

$$C^{(\mathbf{r})}(n, k) = \sum_{i=0}^n \binom{k}{i} \binom{n+2i-1}{3i-1},$$

and the proposition is proved.  $\square$

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